Applied Soft Computing 137 (2023) 110155

Contents lists available at ScienceDirect

Applied Soft Computing

journal homepage: www.elsevier.com/locate/asoc

# A vector visualization of uncertainty complementing the traditional fuzzy approach with applications in project management ☆

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ARTICLE INFO

*Article history:*
Received 14 February 2022
Received in revised form 19 January 2023
Accepted 20 February 2023
Available online 28 February 2023

*Keywords:*
Uncertainty visualization
Triangular fuzzy number
Membership function
Vector representation
Decision making
Project management

ABSTRACT

In view of recent technological developments and new computational and graphical possibilities, scientists and practitioners have become increasingly interested in studying how data and information should be presented. For instance, in project management, it is now recommended to employ dashboards instead of traditional reports. It is also believed that the usage of vectors, resembling the hands of a clock, may increase the efficiency and effectiveness of data presentation and information processing. In light of this, we propose a novel approach to visualization of uncertainty as defined by triangular fuzzy numbers. This new representation is based on vectors whose length represents the range of possible values of an uncertain parameter, while the slope reflects tendencies within possible scenarios. The mathematical foundations and definitions along with the basic properties of this approach are demonstrated in detail. In particular, we show how to transform triangular representations into vector ones and vice versa. The arithmetic operations of addition and multiplication by a crisp number on these vectors are demonstrated as well. Possible applications of the new vector visualization to project uncertainty representation and in project management are described. We also discuss both the advantages and disadvantages of our approach in relation to the traditional visualization as graphs of membership functions. Our proposal is complementary to the traditional one, and they should be used in combination. The new graphical representation of triangular fuzzy numbers expands the available toolbox for visualizing uncertainty not only in project management but in any other area.


## 1. Introduction

Uncertainty, defined on the basic level as "*lack of certainty*" [1, p. 3], is present in everyday life both of individuals and organizations. It is classified in various ways, of which the following stand out [1, p. 33–34], [2, p. 33]:

- Ambiguity or knowledge uncertainty, defined as lack or incompleteness of information. This type of uncertainty depends on the quantity, quality, and relevance of the data and on the reliability and relevance of the models and assumptions.
- Inherent variability uncertainty refers to true differences in attributes due to heterogeneity or diversity. It cannot be reduced by further measurement or study, it can only be better characterized.

Both types of uncertainty can take on different forms [2, p. 36], that is, the scenario, model, and the parameter/input uncertainty. Here, we concentrate on the latter: the parameter uncertainty involved in the specification of numerical values, like cost, duration, etc.

Other authors differentiate between statistical and nonstatistical uncertainty [3]. Statistical uncertainty is the one where probability distributions can be used for quantitative uncertainty analysis. The focus of our research is on nonstatistical uncertainty, where probability distributions are not available. Several approaches have been developed over the years to manage this type of uncertainty. The most popular mathematical models are based on fuzzy [4,5], rough [6], soft [7] and grey sets [8–10]. They are studied and compared to each other in several papers, e.g., [11–13].

In this paper, we focus on triangular fuzzy numbers that can be applied to represent uncertain parameters in various contexts.

☆ **Acknowledgements**: The work of Dorota Kuchta and Jan Schneider was supported by the Polish National Science Centre, under Grant 394311, 2017/27/B/HS4/01881: "Selected methods supporting project management, taking into consideration various stakeholder groups and using type-2 fuzzy numbers". The work of Rafał Michalski was partially financially supported by the Polish National Science Centre under Grant 2019/35/B/HS4/02892: Intelligent approaches for facility layout problems in management, production, and logistics. We thank the four anonymous reviewers for their constructive comments and suggestions on the previous versions of the paper.

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https://doi.org/10.1016/j.asoc.2023.110155
1568-4946/

Apart from uncertainty, the most important problem that lies at the root of the present paper is that of information visualization. Designing appropriate and effective visualizations is far from simple and effortless. As stressed in [14], it cannot be equated with only presenting data on a generated graph, especially if a suitable and insightful graphical representation is crucial for the decision-making process.

Traditionally, fuzzy numbers are represented as graphs of membership functions. In theory, membership functions of any shape can be generated, and many of them may be adequate for specific applications [15]. However, in practice, the vast majority of real-world applications, especially in social, economic, and management sciences, rely on triangular fuzzy numbers. They have been used to model parameter uncertainty in a number of different areas, for example, in supply chain management [16], inventory [17], production planning [18], service quality [19], product positioning [20], decision making [21], project risk estimation [22], project selection [23] or project network analysis [24].

A theoretical explanation for this predominance of triangular fuzzy numbers over all other shapes is given in [25]. The following two arguments may be put forward:

- Simple definition. It requires the estimation of the only three parameters which are used in computing algorithms while intermittent values are usually not involved. This makes triangular fuzzy numbers akin to an enhanced interval number.
- Mathematical tractability. Due to the low complexity of addition and multiplication by a crisp number on the closed set of triangular fuzzy numbers, it is relatively easy to transfer a great number of classical optimization algorithms or procedures from real numbers $\mathbb{R}$ into triangular fuzzy numbers.

Triangular fuzzy number membership functions take the shape of triangles. However, this representation, as simple as it may seem, does not have to be the most appropriate for everyone. For many, the link between the uncertain information and the graphics of triangles may not be obvious. On top of that, certain types of parameter changes in time will be easier to track if they are illustrated by shifts of the tip of a vector, resembling the fluctuations of a clock hand, instead of changes in a triangle shape.

This is especially true in the context of modern project management that, according to both researchers and practitioners, should be based on dashboards instead of traditional reports [26,27]. Dashboards [28] are data visualization and analysis tools that show on one screen all the basic information necessary to make decisions. The idea has resulted in some software applications (e.g., https://thedigitalprojectmanager.com/tools/project-dashboard-software/). Having the appropriate technologies at our disposal, researchers are obliged to investigate modern ways of visualization.

For this reason, in the present work, we propose an alternative vector-based graphical representation of uncertainty as quantified by triangular fuzzy numbers and show its application to modelling data in the context of project management. Because the correspondence between traditional and vector-based representations of triangular fuzzy numbers is unique (one to one), the new representation may be used interchangeably with the traditional one. Since we deal with two alternative representations of the same uncertainty, for the sake of simplicity, we will sometimes call the vectors "*another representation of triangular fuzzy numbers*".

The rest of the paper is organized as follows. In the next section, we briefly provide some specifics about uncertainty modelling and picturing metrics in the context of project management. The following sections include a detailed description of our proposal, with its mathematical foundations and selected properties. We also give some practical examples in the context of project management. The article ends with a discussion of our proposal, conclusions, and some future research prospects.

For convenience, the list of mathematical symbols and abbreviations used in this paper is provided in (Table 1).

## 2. Project management context

Projects can be defined as "*temporary endeavours undertaken to create a unique product, service or result*" [29]. Uniqueness, inherent in projects [30] together with the turbulent project environment and the limited and biased human perception [31], imply that uncertainty is omnipresent in project management. This is particularly true for parameter uncertainty [1, p. 33–34]. Therefore, our research is focused on this uncertainty form, especially in the context of the duration and costs of the project activities.

The uncertainty in projects has to be monitored and controlled, like any other project feature [29]. The knowledge uncertainty is usually the highest at the beginning of projects and diminishes with their advancement. The variability uncertainty, in turn, may be described better and better as the project progresses.

Similarly to other areas, uncertainty modelling in a project can involve fuzzy numbers [32–34]. They have often been used in the project planning stage [35] to model, known only to a certain extent, project task duration times or cost values, as well as other quantitative project parameters. As mentioned in the Introduction, triangular fuzzy numbers are the form of fuzzy numbers that is used most frequently, also in the project management context. Triangular fuzzy numbers express the actual knowledge of experts and their subjective opinion on the pessimistic, most possible, and optimistic value of the respective parameters. This approach has certain advantages over the classical 3-point PERT method of project planning, based on probability theory. Triangular fuzzy numbers do not assume any probability distribution, but refer to the subjective and mathematically simpler possibility theory [5]. This feature makes their application potentially easier for non-mathematicians and non-engineers. Projects are today omnipresent, and they appear not only in companies but also in not-profit organizations, such as non-governmental or public institutions. This makes the problem of understanding and processing information by project participants of all possible backgrounds of utmost importance.

Moreover, projects have been becoming more and more complex [27]. For this reason, project management is nowadays a difficult metaprocess where visualization plays an important role in supporting project managers in their role ([36–38]). Visualization should accelerate the decision-making process and replace text reports. However, an incorrect graphical presentation can lead to improper decisions [39]. That is why it is important to select an adequate way of visualizing uncertain information, adapted to each individual recipient.

The significance of appropriate visualization techniques in conveying information has been recognized in the scientific literature for decades. Some basic rules of human visual processing of graphical data were identified by Gestalt psychologists as early as the beginning of twentieth century [40,41]. They proposed a set of Gestalt laws regarding perceptual grouping, e.g., similarity, closure, proximity, or continuation [42,43]. More recently, among the first approaches to provide guidelines in this regard based on theoretical foundations supported by experimental data, were the works of Cleveland and McGill [44,45]. The authors identified ten elementary perceptual tasks and determined their hierarchy based on the accuracy of the properties compared. They suggested to use those graphical encodings that are as high as

**Table 1**
List of mathematical symbols and abbreviations.

| Symbol | Explanation | First occurrence |
|---|---|---|
| $\xi$ | An uncertain parameter, a fuzzy number | Page 3 |
| $l$ | The smallest possible the uncertain parameter $\xi$ can attain | Page 3 |
| $m$ | The possible value the uncertain parameter $\xi$ can attain | Page 3 |
| $r$ | The largest possible value the uncertain parameter $\xi$ can attain | Page 3 |
| $\xi(x)$ | Membership function of a fuzzy number | Page 3 |
| $tr^*(l, m, r)$ | A triangular fuzzy number in traditional representation | Page 3 |
| $\mu$ | The arithmetic mean of the smallest possible and the largest possible value: $\mu = \frac{\boldsymbol{l+r}}{\boldsymbol{2}}$ | Page 3 |
| $\overrightarrow{vc}(\mu, s, \gamma)$ | A triangular fuzzy number in vector representation | Page 4 |
| $TV$ | Transform taking the traditional representation of a triangular fuzzy number into the vector one | Page 4 |
| $s$ | Length of the vector $\overrightarrow{vc}(\mu, s, \gamma)$: $s = r - l$ | Page 4 |
| $\gamma$ | Angle of deviation from symmetry: $\gamma = \arctan(m - \mu)$ | Page 4 |
| Tail | Cartesian coordinates of the vector initial point in vector representation | Page 4 |
| Tip | Cartesian coordinates of the vector terminal point in vector representation | Page 4 |
| $\gamma_{\triangle min}$ | Smallest possible angle of deviation from symmetry in vector representation | Page 5 |
| $\gamma_{\triangle max}$ | Largest possible angle of deviation from symmetry in vector representation | Page 5 |
| $\underset{\{\triangle\}}{\oplus}$ | Addition operator of triangular fuzzy numbers in traditional representation | Page 6 |
| $\underset{\{\rightarrow\}}{\oplus}$ | Addition operator of triangular fuzzy numbers in vector representation | Page 6 |
| $\underset{\{\sphericalangle\}}{\oplus}$ | Addition operator of angles in the vector representation of triangular fuzzy numbers | Page 6 |
| $\underset{\{\triangle\}}{\odot}$ | Multiplication operator of triangular fuzzy numbers in traditional representation | Page 7 |
| $\underset{\{\rightarrow\}}{\odot}$ | Multiplication operator of triangular fuzzy numbers in vector representation | Page 7 |
| $VT$ | Transform taking the vector representation of a triangular fuzzy number into the traditional one | Page 7 |

possible in the following ordering: position on a common scale and non-aligned scales, length, direction, angle, area, volume, curvature, shading, and colour saturation. In the context of project management, Kerzner [27] provides practical indications concerning the artwork (image, icon) used in the project dashboards, the positioning, accuracy, colour, size, texture, etc. He claims that the selection of graphics used to convey information is extremely important in project management.

Accordingly, our new vector representation of triangular fuzzy numbers extends the set of available techniques for visualizing uncertain information. Therefore, it seems to be potentially interesting and attractive for various stakeholders in project management. The new approach may have some advantages over the traditional triangle representation. The preliminary study related to the usability of vector and membership function-based representations of uncertain data [46] shows its potential usefulness for visualization purposes in the context of project management. In particular, the authors performed an experimentally-based research designed to compare vector and membership function-based representations of uncertainty with respect to their effectiveness, efficiency, and participant satisfaction. They recorded the accuracy of proper recognition of uncertain information presented by both visualizations relative to their textual depiction to objectively assess the effectiveness. The efficiency and user satisfaction measurements were subjectively evaluated by asking about ease of interpretation and attractiveness, respectively. Based on the results from 76 subjects, they found that, overall, both representations appear to be effective to a similar degree in conveying uncertain information. Some significant differences between men and women have been observed. The females performed better if vectors were used, while the males were more effective in the case of traditional representations. Similar patterns were identified for ease of use and attractiveness. Generally, women favoured vector representations, while men preferred traditional representations.

On the whole, the vector representation of fuzzy numbers proposed in [46] was found acceptable and supportive by a substantial number of users. This result prompted us to investigate theoretical aspects regarding soft computing for the vector representation (arithmetical operations), the problem of formal equivalence between the traditional representation of fuzzy numbers and the vector-based one, as well as several mathematical properties of the vector representation. The results of this research are presented here. Also, we focus here specifically on project management context.

A project resembles a car journey. The car driver is heading a destination and controls time and other parameters by means of the dashboard with several meters, that change their indications dynamically. The same is true for a project manager. Here, we discuss, especially in the examples, the dynamic changes in the project situation and compare the respective visualizations (traditional and vector-based one) from the point of view of their user-friendliness. We investigate whether a vector-based dashboard could be, at least for some users, more helpful than the traditional one in decision making during the usually changeable and unstable project implementation process.

No research has been conducted on the difference in the perception of dynamic changes or tendencies in uncertain parameter values represented by triangles and vectors. Here, the above-mentioned clock hand similarity of vectors may be potentially important for facilitating the project monitoring and control for certain project stakeholders.

## 3. Triangular fuzzy numbers and their graphical representations

Triangular fuzzy numbers are a special case of fuzzy sets [5], which are the basis of the possibility theory [5,47]. In economic and social sciences, triangular and trapezoidal fuzzy numbers are by far the most commonly used to represent a quantity whose exact value is not known at the given time [25] and the probability distribution cannot be determined. In this context, the more subjective *possibility degree of occurrence* is used instead of *probability* [48]. The possibility degree is given by experts. The uncertain value of the parameter $\xi$ can be represented by three crisp numbers $l$, $m$ and $r$ such that $l \leq m \leq r$. These numbers correspond to the smallest possible value considered ($l$ as left), the largest possible value ($r$ as right) and the value assumed to be the most possible, i.e. $m$ (middle). The most possible value $m$ has a degree of possibility of 1, $\xi(m) = 1$, whereas the lower and upper bound degrees of possibility are 0, i.e. $\xi(l) = \xi(r) = 0$. A degree of possibility for each $x$: $l \leq x \leq r$ is linearly increasing from

$\xi(l) = 0$ to $\xi(m) = 1$, and linearly decreasing from $\xi(m) = 1$ to $\xi(r) = 0$. This procedure formally defines a so-called *membership function* $\xi(x)$ that, for each real $x$, assigns a degree of possibility according to expert knowledge (1).

$$\xi(x) = \begin{cases} \dfrac{x-l}{m-l} & \text{for } x \in [l, m), \\ 1 & \text{for } x = m, \\ \dfrac{r-x}{r-m} & \text{for } x \in (m, r], \\ 0 & \text{else.} \end{cases} \tag{1}$$

**Notation 1.** *The shorthand notation for (1) used in this paper is presented in (2):*

$$\xi(x) := tr^*(l, m, r)(x), \quad \xi := tr^*(l, m, r). \tag{2}$$

The *support* of $\xi$, as defined in (1), is the closed interval $[l, r]$ [49]. Values between $l$ and $r$ (excluding $l$ i $r$) are considered to be possible to a positive degree. Values beyond this interval are seen as impossible. The broader the interval $[l, r]$, the less information we have about the quantity in question. Its width gives us some information about the indeterminacy degree, which is defined in the Collins dictionary [50] as the quality of being uncertain or vague, and is linked with fuzziness in [51]. The relative position of the value $m$, whose possibility degree is 1, indicates the value which is assumed to be the most possible. Therefore, $m$ shows the skewness of the fuzzy numbers, that is, the inclination of the possibility degree of the values from support $[l, r]$ towards lower (close to $l$), medium (around the middle $\mu = \frac{l+r}{2}$), or higher (close to $r$) values. Let us assume that a neutral situation occurs when the middle value $\mu$ is at the same time the most possible value $m = \mu$. Then, if $m \neq \mu$, the experts have a more pessimistic or optimistic opinion on the possibility distribution. The choice of the adjective depends on the position of $\mu$ with respect to $m$ and on the nature of the value modelled by the fuzzy number. For example, if $(m > \mu)$ and the estimated parameter refers to benefits, the expert opinion will be qualified as optimistic. If $m > \mu$ and the parameter refers to cost, the opinion represented by the same fuzzy number would be pessimistic.

### 3.1. Traditional representation of triangular fuzzy numbers

The hitherto only and generally accepted way to visualize triangular fuzzy numbers is by drawing their membership function $\xi(x)$. Any triangular fuzzy number given in a form $tr^*(l, m, r)$ uniquely determines its membership function (1). For instance, the triangular fuzzy number with $m = 4$ and a scope of indeterminacy $[2, 5]$, denoted as $tr^*(2, 4, 5)$ corresponds to (3) and is visualized by means of the membership function from Fig. 1.

$$tr^*(2, 4, 5)(x) = \begin{cases} \dfrac{(x-2)}{2} & \text{for } x \in [2, 4), \\ 1 & \text{for } x = 4, \\ (5-x) & \text{for } x \in (4, 5], \\ 0 & \text{else.} \end{cases} \tag{3}$$

### 3.2. A new, vector representation of triangular fuzzy numbers

In this paper, we give an alternative representation of the uncertainty defined by triangular fuzzy numbers as vectors pointing upwards from the value $\mu = \frac{l+r}{2}$ and leaning to the left or right side of the vertical line $x = \mu$. For example, the triangular fuzzy number $tr^*(2, 4, 5)$ can be represented as the vector $\overrightarrow{vc}(\mu, s, \gamma) = \overrightarrow{vc}(3.5, 3, 26.55°)$ defined in polar-type coordinates, where $s$ is vector's length, $\gamma$ is the angle between the vector and a vertical line $x = \mu$, and $\mu$ is vector's tail as shown in Fig. 2.

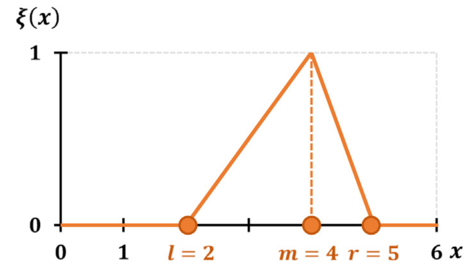


**Fig. 1.** Triangular fuzzy number $tr^*(2, 4, 5)$ represented by a triangle.

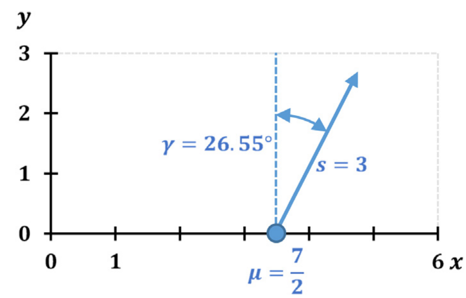


**Fig. 2.** Visualization of triangular fuzzy number $tr^*(2, 4, 5)$ by the vector $\overrightarrow{vc}(3.5, 3, 26.55°)$ attached to $x = \frac{7}{2}$, of length 3, and angle $\gamma = 26.55°$.

In Fig. 2 and subsequent figures a blue dot is placed at $x = \mu = \frac{r+l}{2}$. Vector length $s$ represents the scope of indeterminacy and is equivalent to the length of fuzzy triangle number's support $[l, r]$. The angle $\gamma$, by which the vector is inclined to the left or right, is determined by the relative position of the middle value $\mu$ and the most possible value $m$. The angle indicates that the value $m$ is less than $\mu$ by leaning to the left or greater than $\mu$ – leaning to the right. Thus, the position of the pointer shows clearly the inclination of the represented expert opinion, its pessimistic or optimistic touch.

As a mathematical formula, our new vector representation is defined by the following equivalence transform ($TV$, *Triangle → Vector*), which takes the three defining parameters of a triangular fuzzy number: $l$, $m$, and $r$ into the alternative three parameters $\mu, s, \gamma$ (4).

$$TV: (l, m, r) \to (\mu, s, \gamma) \tag{4}$$

where:

- $\mu$ is the arithmetic mean of $l$ and $r$, $\mu = \frac{r+l}{2}$,
- $s$ is the vector's length equal to the triangle's support length $s = r - l$,
- $\gamma$ is the angle of inclination to the left or right from the vertical line $x = \mu$, which is given by (5).

$$\tan(\gamma) = \frac{m-\mu}{1} \Rightarrow \gamma = \arctan(m-\mu). \tag{5}$$

Thus, the uncertain quantity $\xi$ may be interchangeably and equivalently represented by either the triple $(l, m, r)$ or $(\mu, s, \gamma)$. In the latter case, the notation we adapt is given in (6).

**Notation 2.**

$$\xi := \overrightarrow{vc}(\mu, s, \gamma). \tag{6}$$

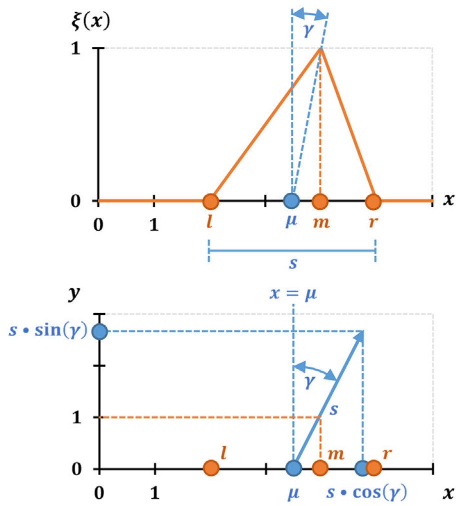


**Fig. 3.** A triangular fuzzy number in its traditional representation in upper graph (orange) and in a vector form in down graph (light blue).

In compact form the transform $TV$ of one defining triple to the other is given by (7):

$$TV\left[tr^{*}(l,m,r)\right] \mapsto \overrightarrow{vc}\left(\frac{l+r}{2}, r-l, \arctan\left(m-\frac{l+r}{2}\right)\right) := \overrightarrow{vc}\,(\mu,s,\gamma) \tag{7}$$

The geometric meaning and provenance of $\mu$, $s$, and $\gamma$ is shown in Fig. 3, where we can also compare both representations. The vector coefficients (parameters) can be identified graphically in the traditional representation as is presented in the graph of its membership function.

The vector representation of triangular fuzzy numbers defined in polar coordinates, can also be expressed by Cartesian coordinates, which may, at times, be more convenient to use. The vector's tip and tail coordinates of the given triple $(\mu,\ s,\ \gamma)$ can be computed according to (8), (9), and (10).

$$\text{Tail}\,(\mu,0)\ \text{ and tip}\,(\mu+s\cdot\sin(\gamma)\,,\ s\cdot\cos(\gamma))\,, \tag{8}$$

whereby

$$\sin(\gamma) = \frac{m-\mu}{\sqrt{1^2+(m-\mu)^2}}, \tag{9}$$

and

$$\cos(\gamma) = \frac{1}{\sqrt{1^2+(m-\mu)^2}}. \tag{10}$$

**Example 1.** Tip and tail vector representation of a triangular fuzzy number.

The triangular fuzzy number $tr^{*}(1,\ 4,\ 5)$ may be written in a vector representation as (11).

$$\overrightarrow{vc}\ (3,\ 4,\ 45^\circ)\,, \tag{11}$$

or as tip and tail coordinates by (12).

$$\text{Tail} = \begin{pmatrix}3\\0\end{pmatrix},\ \text{tip} = \begin{pmatrix}5.83\\2.83\end{pmatrix}, \tag{12}$$

as shown in Fig. 4.

### *3.3. Border cases of the vector representation*

Theoretically, in the general case, $\gamma$ can take values from $-90^\circ$ and $+90^\circ$ as given in (13), (14), and (15) and schematically

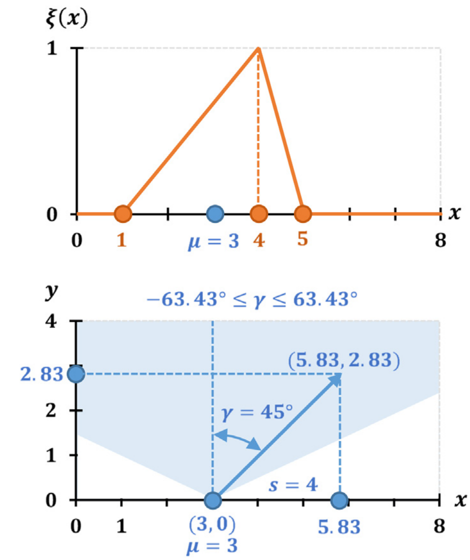


**Fig. 4.** The traditional and vector representations corresponding to the triangular fuzzy number $tr^{*}(1,\ 4,\ 5)$ given in Cartesian coordinates with indicated tail and tip coordinates.

presented in Fig. 5.

$$\gamma_{\min} = \lim_{(m-\mu)\to-\infty} \arctan(m-\mu) \to -90^\circ, \tag{13}$$

$$\gamma_{\max} = \lim_{(m-\mu)\to\infty} \arctan(m-\mu) \to 90^\circ, \tag{14}$$

that is to say that:

$$\gamma \in (-90^\circ,\ +90^\circ)\,. \tag{15}$$

However, the triangular fuzzy number fulfils in fact the condition $\gamma \in \left[\gamma_{\triangle min}, \gamma_{\triangle max}\right]$, where the interval of possible $\gamma$ values is a proper subset of the interval $(-90^\circ, +90^\circ)$. Since $\mu$ is the *middle* point of the support of the triangle $tr^{*}(l,\ m,\ r)$, $(m-\mu)$ measures the degree of deviation from symmetry meant as $m=\mu=\frac{r+l}{2}$. One may single out three border cases that determine the value range of $\gamma$. For symmetric triangular fuzzy numbers, we have (16):

$$\mu = m \to \gamma_{\triangle sym} = \arctan(m-\mu) = \arctan(0) = 0^\circ \tag{16}$$

The other two cases refer to the so-called degenerate triangle numbers of type $tr^{*}(m,\ m,\ r)$ or $tr^{*}(l,\ m,\ m)$. This time the most possible value $m$ coincides with one of the endpoints of the support $[l,\ r]$, and then goes to infinity with the left or right endpoint. Therefore, according to (13) and (14), one receives an angle of $\gamma=-90^\circ$ or $+90^\circ$. Maximal and minimal values of $\gamma$ depend on maximal and minimal values of $(m-\mu)$ and can be computed by (17) and (18) respectively.

$$\min(m-\mu) \to m=l \to m-\mu = l-\frac{r+l}{2} = \frac{2l-r-l}{2}$$
$$= -\frac{r-l}{2} = -\frac{s}{2}, \tag{17}$$
$$\gamma_{\triangle min} = \arctan\left(-\frac{s}{2}\right)$$

$$\max(m-\mu) \to m=r \to m-\mu = r-\frac{r+l}{2} = \frac{2r-r-l}{2}$$
$$= \frac{r-l}{2} = \frac{s}{2},$$
$$\gamma_{\triangle max} = \arctan\left(\frac{s}{2}\right) \tag{18}$$

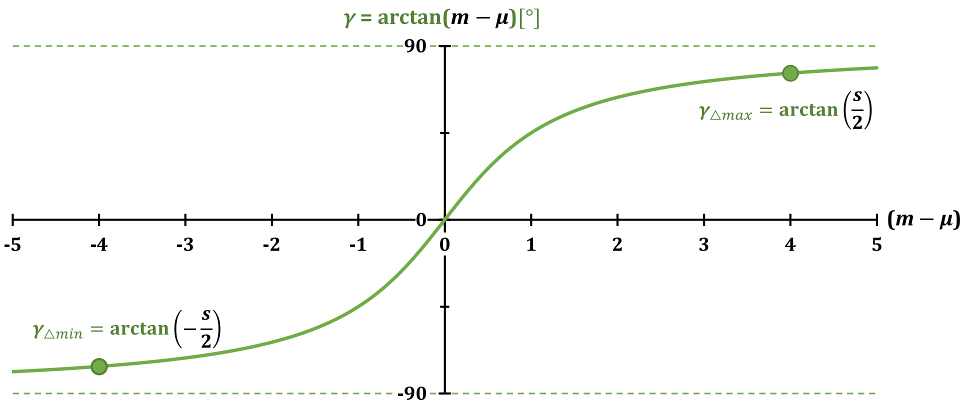


**Fig. 5.** Visualization of $\gamma$ maximal and minimal values in general, and in triangular fuzzy numbers for an exemplary $s = 8$, and $\gamma_{\triangle min} = -75.96^{\circ} < \gamma < +75.96^{\circ} = \gamma_{\triangle max}$, marked with the green dot.

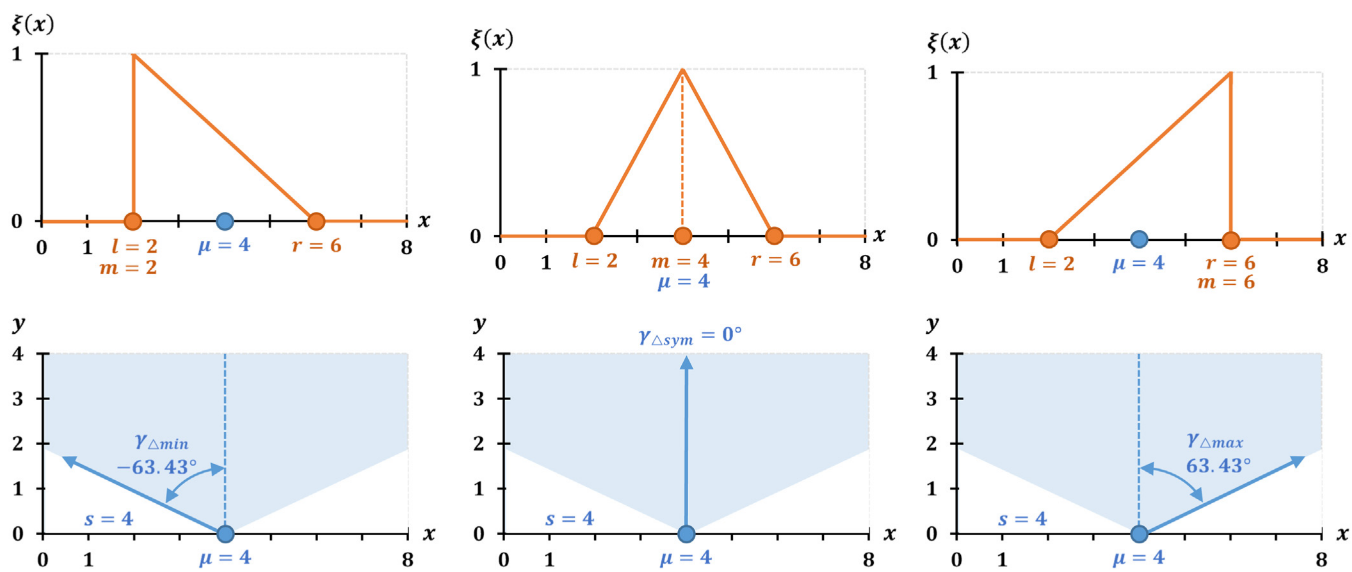


**Fig. 6.** Symmetricity and degeneracies in traditional and vector representations of the uncertainty defined by (1).

Thus, for degenerate triangular fuzzy numbers, the $(m - \mu)$ value ranges from $-\frac{r-l}{2}$ to $+\frac{r-l}{2}$ and extreme values of $\gamma$ depend only on the indeterminacy degree $s$. One can also observe that the higher the indeterminacy measured by $s$, the higher maximal possible absolute values of $\gamma$ are. For example, for any triangular fuzzy numbers such that $s = r - l = 4$ and $r \geq m \geq l$, $\gamma \in (-63.43^{\circ}, +63.43^{\circ})$. Sample cases of degenerative triangular fuzzy numbers with $s = 4$, i.e. $tr^{*}(2, 2, 6)$, $tr^{*}(2, 4, 6)$, and $tr^{*}(2, 6, 6)$ are shown in Fig. 6. The light blue area in the vector representations shows the possible range of $\gamma$ values for these examples.

### *3.4. Arithmetical operations in traditional and vector representations*

In this section, we present the addition of two triangular fuzzy numbers and the multiplication of a triangular fuzzy number by positive crisp numbers in both representations. These operations are sufficient for most project management applications, e.g., the determination of project budgets, schedules, or risk appraisal.

**Notation 3.** *In this paper, we denote addition in traditional representation by* $\underset{\{\triangle\}}{\oplus}$ *and addition in the vector representation by* $\underset{\{\rightarrow\}}{\oplus}$.

Before defining the addition of triangular fuzzy numbers in their vector representation, we provide a reminder of the addition definition for membership functions. For two triangular fuzzy numbers given in their traditional representation $tr^{*}(l_1, m_1, r_1)$ and $tr^{*}(l_2, m_2, r_2)$ one has by definition (19) [4].

$$tr^{*}(l_1, m_1, r_1) \underset{\{\triangle\}}{\oplus} tr^{*}(l_2, m_2, r_2) = tr^{*}(l_1 + l_2,\ m_1 + m_2,\ r_1 + r_2)\,. \tag{19}$$

The addition operation in the vector representation for two vectors $\overrightarrow{vc}(\mu_1, s_1, \gamma_1)$ and $\overrightarrow{vc}(\mu_2, s_2, \gamma_2)$ must be mathematically consistent with (19), which necessarily entails definitions (20) and (21). For easier readability, we first introduce the following addition operation on angles.

**Notation 4.** *In this paper, we denote the addition of angles in the vector representation of triangular fuzzy numbers by* $\underset{\{\sphericalangle\}}{\oplus}$.

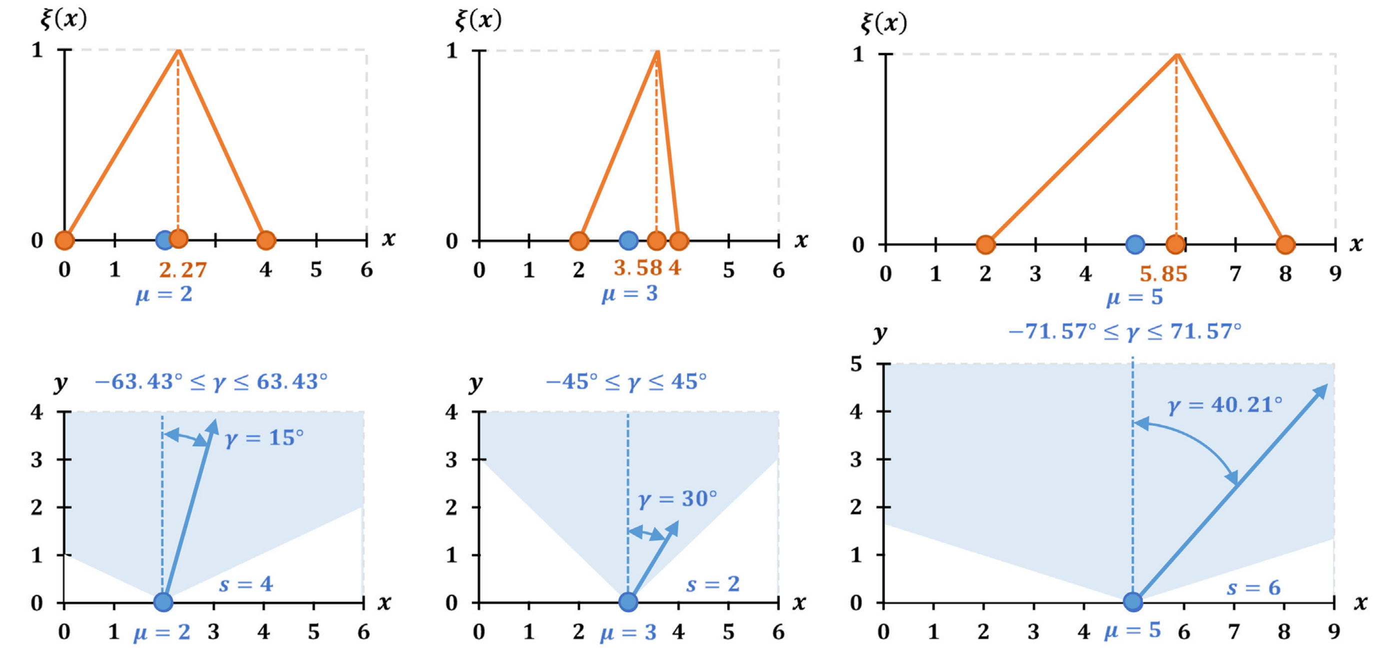


**Fig. 7.** Addition of two triangular fuzzy numbers in traditional and vector representations.

**Definition 1.**

$$\gamma_1 \underset{\{\triangleleft\}}{\oplus} \gamma_2 = \arctan\left(\tan\left(\gamma_1\right) + \tan\left(\gamma_2\right)\right) \tag{20}$$

**Definition 2.**

$$\vec{vc}\left(\mu_1, s_1, \gamma_1\right) \underset{\{\rightarrow\}}{\oplus} \vec{vc}\left(\mu_2, s_2, \gamma_2\right) = \vec{vc}\left(\mu_1 + \mu_2, s_1 + s_2, \gamma_1 \underset{\{\triangleleft\}}{\oplus} \gamma_2\right) \tag{21}$$

**Remark 1.** The vectors given in (8) cannot be added in the usual $\mathbb{R}^2$ sense because of (22) and (23):

$$\sin\left(\gamma_1 \underset{\{\triangleleft\}}{\oplus} \gamma_2\right) = \frac{\left(m_1 + m_2\right) - \left(\mu_1 + \mu_2\right)}{\sqrt{1^2 + \left(\left(m_1 + m_2\right) - \left(\mu_1 + \mu_2\right)\right)^2}}$$
$$\neq \frac{m_1 - \mu}{\sqrt{1^2 + \left(m_1 - \mu_1\right)^2}} + \frac{m_2 - \mu}{\sqrt{1^2 + \left(m_2 - \mu_2\right)^2}} = \sin\left(\gamma_1\right) + \sin\left(\gamma_2\right) \tag{22}$$

$$\cos\left(\gamma_1 \underset{\{\triangleleft\}}{\oplus} \gamma_2\right) = \frac{1}{\sqrt{1^2 + \left(\left(m_1 + m_2\right) - \left(\mu_1 + \mu_2\right)\right)^2}}$$
$$\neq \frac{1}{\sqrt{1^2 + \left(m_1 - \mu_1\right)^2}} + \frac{1}{\sqrt{1^2 + \left(m_2 - \mu_2\right)^2}} = \cos\left(\gamma_1\right) + \cos\left(\gamma_2\right). \tag{23}$$

For instance (24):

**Example 2.** Adding two vector representations of triangular fuzzy numbers.

$$\vec{vc}\ (2,\ 4,\ 15°) \underset{\{\rightarrow\}}{\oplus} \vec{vc}\ (3,\ 2,\ 30°) = \vec{vc}\ (5,\ 6,\ 40.21°) \tag{24}$$

as depicted in Fig. 7.

Some more additions of sample vector representations are given in Example 3 (25). They serve mainly to give a feeling for the $TV$-addition of angles, as the first and second components of the triples added in (21) are straightforward.

**Example 3.** Additions of sample vectors.

$$\begin{aligned}
&\vec{vc}\ (3, 2, 30°) \underset{\{\rightarrow\}}{\oplus} \vec{vc}\ (5, 1, 10°) = \vec{vc}\ (8, 3, 37.00°)\\
&\vec{vc}\ (3, 2, -30°) \underset{\{\rightarrow\}}{\oplus} \vec{vc}\ (5, 1, -10°) = \vec{vc}\ (8, 3, -37.00°)\\
&\vec{vc}\ (3, 2, 50°) \underset{\{\rightarrow\}}{\oplus} \vec{vc}\ (5, 1, 60°) = \vec{vc}\ (8, 3, 71.12°)\\
&\vec{vc}\ (3, 2, 50°) \underset{\{\rightarrow\}}{\oplus} \vec{vc}\ (5, 1, -60°) = \vec{vc}\ (8, 3, -28.38°)\\
&\vec{vc}\ (3, 2, -50°) \underset{\{\rightarrow\}}{\oplus} \vec{vc}\ (5, 1, -60°) = \vec{vc}\ (8, 3, -71.12°)
\end{aligned} \tag{25}$$

**Remark 2.** In general, we have, by construction, (26):

$$-90° < \gamma_{\triangle min} < \gamma < \gamma_{\triangle max} < +90°. \tag{26}$$

which can be schematically seen in Fig. 5. For consistency, Definition 1 (20) must yield (27):

$$-90° < \left(\gamma_1 \underset{\{\triangleleft\}}{\oplus} \gamma_2\right)_{\triangle min} \leq \left(\gamma_1 \underset{\{\triangleleft\}}{\oplus} \gamma_2\right) \leq \left(\gamma_1 \underset{\{\triangleleft\}}{\oplus} \gamma_2\right)_{\triangle max} < +90°, \tag{27}$$

Assuming (28):

$$\gamma_1 = \arctan\left(m_1 - \mu_1\right), \quad \gamma_2 = \arctan\left(m_2 - \mu_2\right), \tag{28}$$

we observe that (29):

$$\begin{aligned}
\gamma_1 \underset{\{\triangleleft\}}{\oplus} \gamma_2 &= \arctan\left(\tan\left(\gamma_1\right) + \tan\left(\gamma_2\right)\right)\\
&= \arctan\left(\left(m_1 - \mu_1\right) + \left(m_2 - \mu_2\right)\right)\\
&= \arctan\left(\left(m_1 + m_2\right) - \left(\mu_1 + \mu_2\right)\right).
\end{aligned} \tag{29}$$

Let us add, for instance, two triangular fuzzy numbers with a large indeterminacy of $s = 115$, e.g., $tr^*$ (0, 115, 115). The maximal vector angle for these individual cases is $\gamma_{\triangle \max} = 89°$. The maximal $\gamma_{\triangle \max}$ for the resulting vector $tr^*$ (0, 230, 230) is 89.5° (30).

$$89^\circ \underset{\{\triangleleft\}}{\oplus} 89^\circ = \arctan(\tan(89^\circ) + \tan(89^\circ)) \simeq 89.5^\circ. \tag{30}$$

This specific borderline example shows that the sensitivity of changing the vector direction is smaller for larger absolute values of $\gamma$, that is while deviating farther from the vertical line $x = \mu$.

**Notation 5.** *In this paper, we denote the product of a positive crisp real number and a triangular fuzzy number in the traditional representation by* $\underset{\{\triangle\}}{\odot}$ *and the equivalent for the vector representation by* $\underset{\{\rightarrow\}}{\odot}$.

For any real positive crisp value $c$ and a triangular fuzzy number with non-negative $l, m, r$, given in its traditional representation $tr^*(l, m, r)$ one has by definition (31).

$$c \underset{\{\triangle\}}{\odot} tr^*(l, m, r) = tr^*(c \cdot l, c \cdot m, c \cdot r). \tag{31}$$

The equivalent formula of (31) in the vector representation of the triangular fuzzy number is given in (32).

$$c \underset{\{\rightarrow\}}{\odot} \overrightarrow{vc}(\mu, s, \gamma) = \overrightarrow{vc}(c \cdot \mu, c \cdot s, \arctan(c \cdot \tan(\gamma))). \tag{32}$$

Formula (32) results from transformation $TV$ (7) in the following way (33).

$$\begin{aligned} & TV\left[tr^*(c \cdot l, c \cdot m, c \cdot r)\right] \\ & = \overrightarrow{vc}\left(\frac{c \cdot l + c \cdot r}{2}, c \cdot r - c \cdot l, \arctan\left(c \cdot m - \frac{c \cdot l + c \cdot r}{2}\right)\right) \\ & = \overrightarrow{vc}\left(c \cdot \frac{l + r}{2}, c \cdot (r - l), \arctan\left(c \cdot \left(m - \frac{l + r}{2}\right)\right)\right) \\ & = \overrightarrow{vc}(c \cdot \mu, c \cdot s, \arctan(c \cdot (m - \mu))) \\ & = \overrightarrow{vc}(c \cdot \mu, c \cdot s, \arctan(c \cdot \tan(\gamma))) \end{aligned} \tag{33}$$

For example, $2 \underset{\{\triangle\}}{\odot} tr^*(0, 2, 3) = tr^*(0, 4, 6)$ is equivalent with $2 \underset{\{\rightarrow\}}{\odot} \overrightarrow{vc}(1.5, 3, 26.57^\circ) = \overrightarrow{vc}(3, 6, 45^\circ)$. A graphical illustration of this example is shown in Fig. 8. It should be noted that the multiplication of the triangular fuzzy number by a crisp positive real value does not change the sign of the $\gamma$ angle in its vector representation.

### 3.5. Back-transforming from vectors to triangles

With (7) showing the transformation from traditional representation to vectors, it is necessary to be able to go back. That is, given vector representation $\overrightarrow{vc}(\mu, s, \gamma)$, compute the corresponding traditional representation $tr^*(l, m, r)$. Obviously, $l = \mu - \frac{s}{2}$, and $r = \mu + \frac{s}{2}$. The $m$ parameter can be obtained from the fact that $\tan(\gamma) = m - \mu$ (see (5)), which gives us $m = \tan(\gamma) + \mu$. Thus, this back-transformation $TV^{-1} = VT$ from vector to traditional representation is given in (34):

$$VT\left[\overrightarrow{vc}(\mu, s, \gamma)\right] = tr^*\left(\mu - \frac{s}{2}, \tan(\gamma) + \mu, \mu + \frac{s}{2}\right) = tr^*(l, m, r). \tag{34}$$

It may be desirable to perform computations in one representation (triangles) but display results in the other (vectors). Therefore, it needs to be ensured that at least additive operations carry over from one representation to the other and back. That is, we need to show (35):

$$TV\left[tr_1^* \underset{\{\triangle\}}{\oplus} tr_2^*\right] = TV\left[tr_1^*\right] \underset{\{\rightarrow\}}{\oplus} TV\left[tr_2^*\right]. \tag{35}$$

This is shown by performing the following calculations (36).

$$\begin{aligned} & TV\left[tr_1^* \underset{\{\triangle\}}{\oplus} tr_2^*\right] = TV\left[tr^*(l_1 + l_2, m_1 + m_2, r_1 + r_2)\right] \\ & = \overrightarrow{vc}\left(\frac{(r_1 + r_2) - (l_1 + l_2)}{2}, (r_1 + r_2)\right. \\ & \left. - (l_1 + l_2), \arctan\left[m_1 + m_2 - \frac{(r_1 + r_2) - (l_1 + l_2)}{2}\right]\right) \\ & = \overrightarrow{vc}\left(\frac{r_1 - l_1}{2} + \frac{r_2 - l_2}{2}, (r_1 - l_1)\right. \\ & \left. + (r_2 - l_2), \arctan\left[\left(m_1 - \frac{r_1 + r_2}{2}\right) + \left(m_2 - \frac{l_1 + l_2}{2}\right)\right]\right) \\ & = \overrightarrow{vc}(\mu_1 + \mu_2, s_1 + s_2, \arctan(\tan(\gamma_1) + \tan(\gamma_2))) \\ & = \overrightarrow{vc}(\mu_1, \gamma_1, s_1) \underset{\{\rightarrow\}}{\oplus} \overrightarrow{vc}(\mu_2, \gamma_2, s_2) = TV\left[tr_1^*\right] \underset{\{\rightarrow\}}{\oplus} TV\left[tr_2^*\right]. \end{aligned} \tag{36}$$

## 4. Examples of applications to project management

In this section, we illustrate the usage of both representations of the uncertainty represented by triangular fuzzy numbers in the context of project management. In project management, one distinguishes 10 project management knowledge areas and 49 project management process groups [52]. Our examples relate mainly to the knowledge area of Project Schedule Management and to two process groups linked to this knowledge area: Estimate Activity Durations and Control Schedule, as well as to the knowledge area of Project Risk Management.

The example fuzzy numbers which are displayed in Figs. 9, 10, and 11 represent durations of not yet started project activities, which are characterized by non-statistical uncertainty.

A possible scenario served by the examples is the following. There are three potential suppliers of a service which is indispensable for the activity (for instance, providing a number of hoists and equipment handling at some stage of a construction project) whose time duration is being estimated. The first supplier's service is superior to other suppliers' service, resulting in a maximally shortened activity performance time. The activity duration corresponding to this supplier is represented by the number $l$. Due to the supplier's high financial demands, it is hardly possible that this firm will be contracted. Then there is another supplier, supplier 3, whose service is inferior, resulting in a prolonged duration of the activity, denoted as $r$. Finally, there is a compromise supplier 2 — the time we estimate they will need for the job is $m$. Most possibly, this supplier will be contracted; therefore, the degree of possibility is 1.

A change in the internal situation of each of these suppliers (financial trouble, unexpected sick leaves, defaults of other clients, or on the contrary, a sudden increase in productivity) can affect each of the three estimates of $l, m, r$ independently — as the project proceeds, each of the estimates may have to be moved in one or other direction.

A different scenario serving the same example Figs. 9, 10, and 11 could be a single contractor for the same activity who has a varying (maximal, minimal, in between) workforce.

In the context of time duration of activities in a project the optimistic inclination would mean that the most possible value $m$ is closer to the minimal possible value $l$ and $\gamma$ negative, the pessimistic inclination ($\gamma$ positive) would occur in the opposite case. The fuzzy estimate of an activity duration directly influences the predicted entire project duration, if the task belongs to the critical path [39].

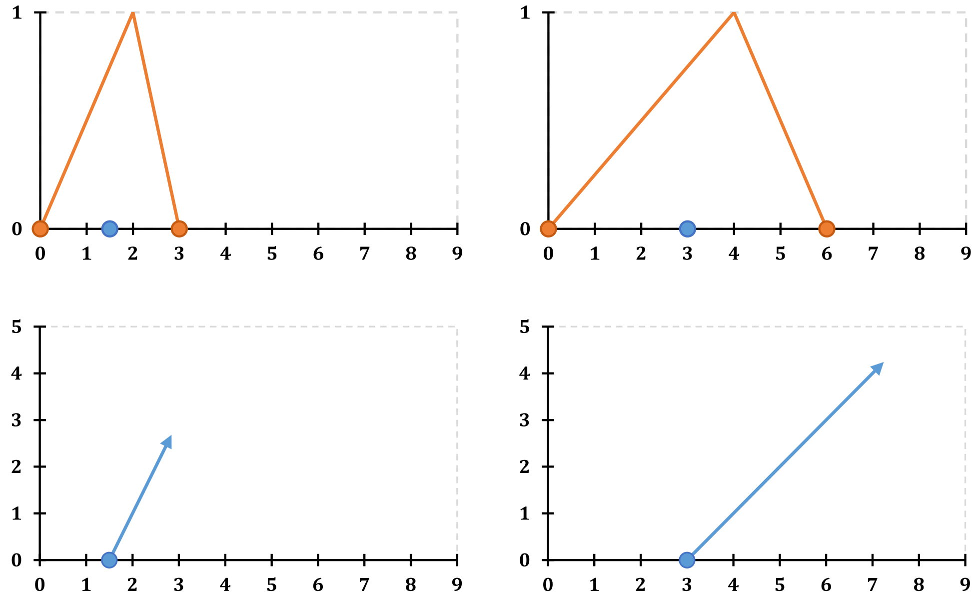


**Fig. 8.** Multiplication of the triangular fuzzy number by a crisp positive real value in traditional $\left(2 \underset{\{\triangle\}}{\odot} tr^*(0, 2, 3) = tr^*(0, 4, 6)\right)$, and vector $\left(2 \underset{\{\to\}}{\odot} \overrightarrow{vc}(1.5, 3, 26.57^\circ) = \overrightarrow{vc}(3, 6, 45^\circ)\right)$ representations.

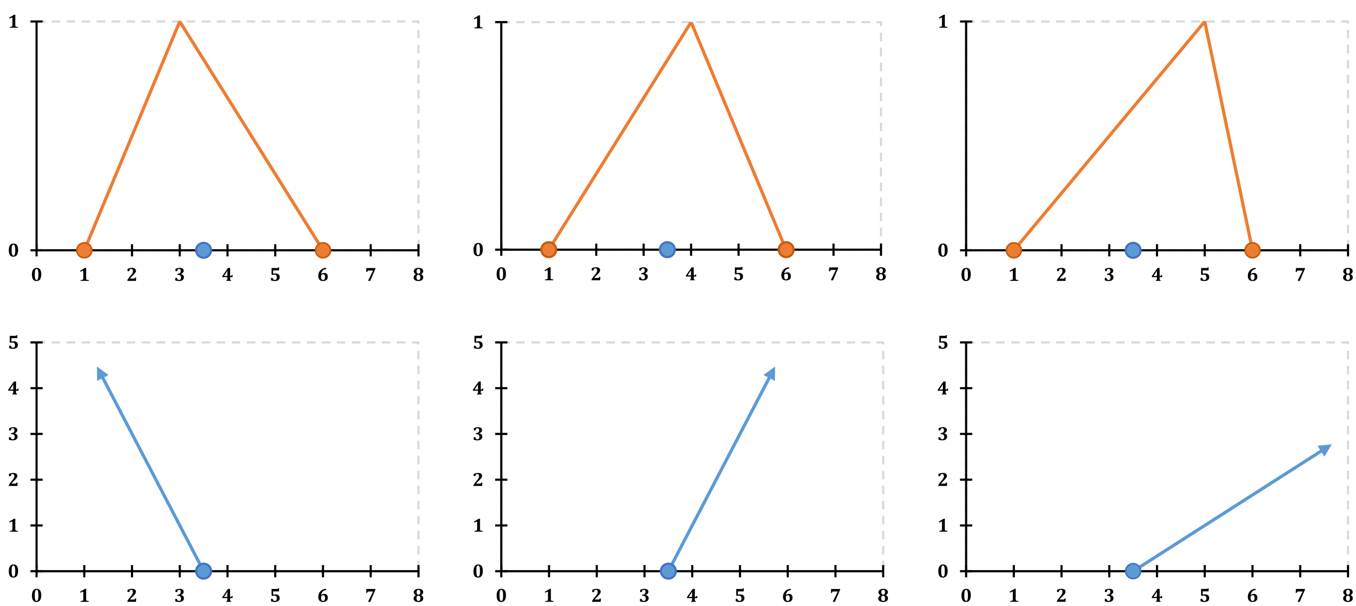


**Fig. 9.** Estimation of the task duration changing over time in Example 4. From rather optimistic assessment to gradually more and more pessimistic.

The selection of experts and the entire Estimate Activity Durations process are described in detail in [53]. The process is usually based on interviews with people who are knowledgeable about the project or its selected aspects. They are called "*subject matter experts*" and are most often members of the project team and its management, contractors, or advisors. The type and structure of interviews must be adapted to the interviewee and the information required. They have to take into account various types of biases that deteriorate the estimation quality. Sometime estimates are developed in workshops with 20–25 participants rather than interviews with a single or a few persons. Three-point estimates are especially popular. In our examples, we assumed arbitrary triangular fuzzy values that were selected to represent some of the typical scenarios in the project management practice.

The estimations of activity durations are performed in the following moments:

– Before the project starts with respect to all project activities or tasks – the two notions will be treated as synonyms. Various duration estimation methods mainly based on expert knowledge can be used here [52].
– Repeatedly at constant intervals during the whole project course with respect to yet non-started project activities. Here, updated expert knowledge and experience gathered in the project so far, is used.

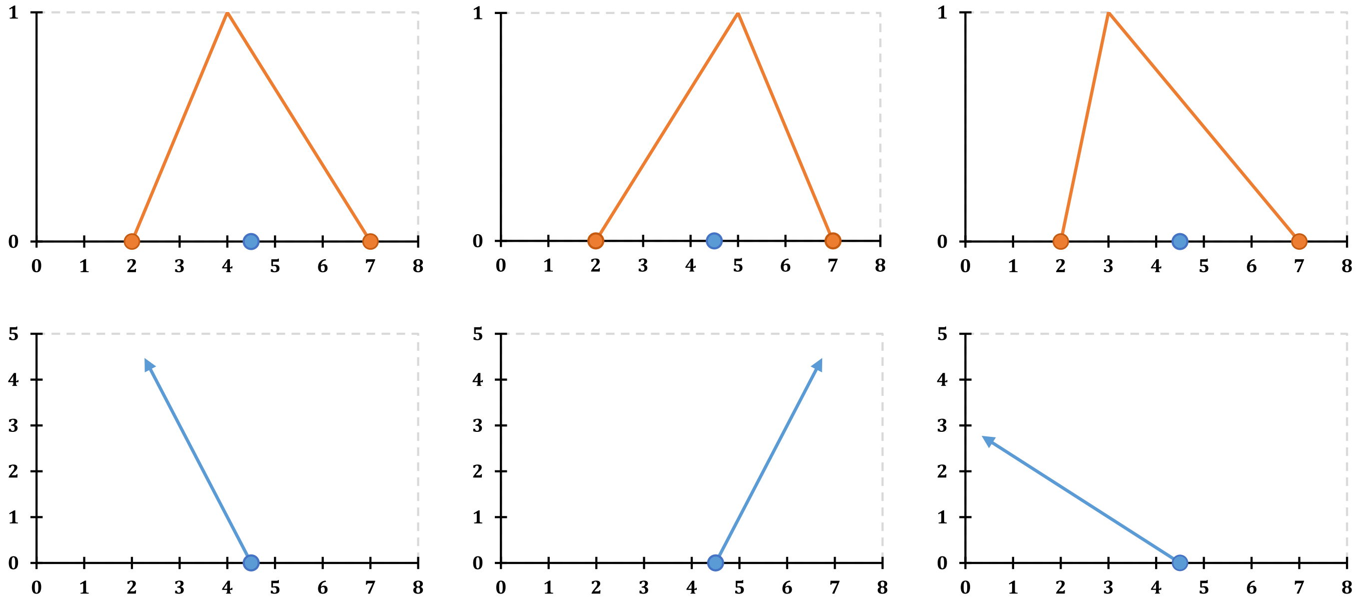

**Fig. 10.** Estimation of task duration changing over time in Example 5. From positive assessment to negative and again to even more positive.

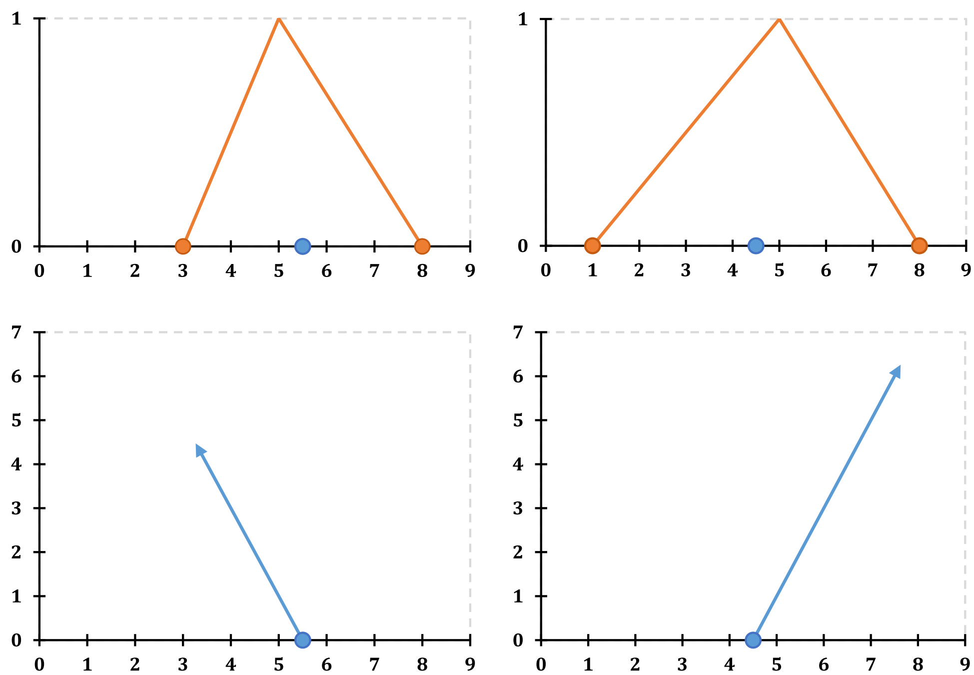

**Fig. 11.** Two different tasks at the same time in traditional and vector representations.

In the first two examples (Examples 4 and 5) we consider three consecutive moments of the project and one and the same activity, which has not been started up to the latest of those moments. The activity duration is expressed, in each of the considered moments, by means of triangular fuzzy numbers provided by experts. The first moment $t_0$ is the project start, the two other moments occur during project execution.

**Example 4.** Duration estimation of a task updated by experts in consecutive moments. From a rather optimistic assessment to a gradually more and more pessimistic assessment.

Here (Table 2) the experts are convinced at all the three moments of time $t_i, i = 0 \ldots 2$ that the task duration $x$ would be included somewhere in the interval $x \in [1, 6]$. However, with the project advancing, and additional information available, they change their mind about the inclination (skewness) of the fuzzy estimation, i.e. the relative position of the most possible value with respect to the support centre. At the beginning ($t_0$), they think the most possible value of the task duration is about $m = 3$, thus, they assign to the fuzzy estimation an optimistic inclination. Later, at time $t_1$, it is re-evaluated to $m = 4$ changing the estimation inclination in the pessimistic direction, and at moment $t_2$, the experts' opinion shifts to $m = 5$, thus the pessimistic inclination increases. This means that as the project progresses, the estimation of the most possible value of the task duration turns out to be more pessimistic than it was judged at the project beginning, and this tendency is persevering. Furthermore, at a

**Table 2**
The task duration uncertainty estimated at three moments in time for Example 4, given in traditional and vector representations.

| Type of representation | Task duration estimates | | |
|---|---|---|---|
| | $t_0$ | $t_1$ | $t_2$ |
| Triangular | $tr^*$ (1, 3, 6) | $tr^*$ (1, 4, 6) | $tr^*$ (1, 5, 6) |
| Vector | $\overrightarrow{vc}$ (3.5, 5, −26.57°) | $\overrightarrow{vc}$ (3.5, 5, +26.57°) | $\overrightarrow{vc}$ (3.5, 5, +56.31°) |

**Table 3**
The task duration uncertainty estimated at three moments in time for Example 5, given in traditional and vector representations.

| Type of representation | Task duration estimates | | |
|---|---|---|---|
| | $t_0$ | $t_1$ | $t_2$ |
| Triangular | $tr^*$ (2, 4, 7) | $tr^*$ (2, 5, 7) | $tr^*$ (2, 3, 7) |
| Vector | $\overrightarrow{vc}$ (4.5, 5, −26.57°) | $\overrightarrow{vc}$ (4.5, 5, +26.57°) | $\overrightarrow{vc}$ (4.5, 5, −56.31°) |

certain moment in time the optimism of experts with respect to the duration of the task in question turns into pessimism. Such pieces of information are crucial for the Control Schedule and Project Risk Management processes. The project manager and the project team should address the problem as soon as possible, preferably before the task in question starts. The sooner they react, the more chances they have to solve the issue: either by reducing the increasing most possible task duration or by making sure that the longer most possible duration will be acceptable from the point of view of the final project evaluation. The tendency observed in the task estimation may also indicate some serious problems in the project, and the sooner they are identified, the higher the project success chances.

Fig. 9 shows a graphical confrontation of the traditional and vector representations of this example. Let us compare these visualizations with respect to the basic information they should convey, that is, the increasing most possible duration of the task in question and the switch from an optimistic opinion on this value to a more and more pessimistic one. This is shown by the shift of the most possible duration value of the activity in question from below the middle value of the interval $[1, 6]$ to farther and farther to the right. The visualization should be as striking as possible to attract the attention of the, usually very busy, authorized members of the team, at the earliest possible moment. In the authors' opinion, the vector representation works in this respect better than the traditional one. The traversal of the middle point of interval $[1, 6]$ is much more clearly visible in the case of vectors, as the pointer completely changes the direction.

Also, if we consider the S-type shape of the function in Fig. 5, we can see that the tendency which started before moment $t_1$ (the gradual passage from optimistic inclination in moment $t_0$ to a deepening pessimism) will be most visible shortly before and right after the traversal of the middle point. In this area, small changes of $m - \mu$ influence the most the value of $\gamma$. This means that the vector representation is most appealing (the clock hand moves the most) around the neutral point $m - \mu$: shortly before it and right after it. Although the changes in the triangles' shapes show the same moves of inclination from optimism to pessimism, they do so by far less clearly, thus, seem to be less useful from the point of view of effective project management.

**Example 5.** Duration estimation of a task updated by experts in consecutive moments. From rather optimistic assessment to a pessimistic one, and back to more optimistic that the original one.

In this example, we take a look at a similar situation as in Example 4 but the tendency in the task duration estimations is different. Here, the triangular fuzzy numbers are defined as in Table 3.

Graphical visualizations of both representations are shown in Fig. 10. The changes, as in the previous example, regard the most possible values. However, here, after the negative shift from below the middle point of the interval $[2, 7]$ ($t_0$) to a value over the middle point in moment $t_1$, the estimation of the most possible value in moment $t_2$ returns below the middle value and is better (lower) than in moment $t_0$. Thus, we have a shift from optimism to pessimism and then back to even higher optimism than originally. This could be achieved thanks to taking some decisive measures addressing the initially negative trend, after the warning received in moment $t_1$. Again, the interpretation of the three estimations of parameter uncertainty of this trend and the switch between optimism and pessimism appears to be clearer in the vector representation compared to the traditional approach. The changes in the positions of *clock hands* are more appealing than the changes in the shape of the triangles.

**Table 4**
Estimations of the duration of a project task in two modes A and B for Example 6, given in traditional and vector representations.

| Type of representation | Task duration estimates ($t_0$) | |
|---|---|---|
| | Mode A | Mode B |
| Triangular | $tr^*$ (3, 5, 8) | $tr^*$ (1, 5, 8) |
| Vector | $\overrightarrow{vc}$ (5.5, 5, −26.57°) | $\overrightarrow{vc}$ (4.5, 7, +26.57°) |

In Example 6 we consider one single moment, before the project starts, in which we estimate the duration of a task. In this moment, two different execution modes are possible, e.g., executed in two distinct technologies or by two separate teams. We should choose one mode for actual execution.

**Example 6.** Comparison of two different task modes at the same time, before the project starts.

Let us now consider a single moment $t_0$ before the project starts, and a project task with duration estimations of task execution modes A and B, specified as in Table 4.

Here we are facing a situation where the duration estimations for both modes have the same most possible value ($m = 5$), but its position with respect to the middle value of the supports along with the lengths of the supports are different. These representations are visualized in Fig. 11.

In this example, the duration of the task in both modes has the same most possible value. Thus, the selection of the mode has to be performed using other criteria. The natural proposals are as follows:

I. the minimal possible value — preferably as small as possible,
II. the maximal possible value — preferably as small as possible,
III. the indeterminacy degree — preferably as small as possible,
IV. the optimism degree — preferable as big as possible.

The decision maker has to select the criteria and their relative importance to make the final decision. For the evaluation of the modes under criteria I and II, the decision maker will probably be better supported by the traditional representation. However, for criteria III and IV, the vector representation appears to be more efficient and effective. The vector position immediately shows the inclination, either pessimistic or optimistic, whereas the vector length permits one to evaluate easily the indeterminacy degree. Although the same information can be extracted from the traditional representation, this would require from the decision maker much more mental effort.

Similar images as in Figs. 9–11 could be displayed on the project dashboard for the total duration of the project, estimated in the given moment and calculated as the sum of activity durations from the critical path, thus the longest project network path. Since in the fuzzy case the longest path is not always uniquely determined (see, e.g., [54–57]), the length estimates and the consecutive changes for several paths might have to be displayed on one dashboard. This means that, depending on the representation chosen, the decision maker would be presented either with several triangles or with several vectors. The vector-based display that uses "clock-similar" symbols can be, in our opinion, more user-friendly than a display based on triangles.

## 5. Discussion and conclusions

In this research, we proposed a novel method for visualizing uncertainty. In this new approach, we use vectors to graphically show uncertain information about a given parameter, which is traditionally represented by triangular membership functions. Our vectors are defined by three crisp numbers ($\mu$, $s$, $\gamma$). The traditional representation also uses three parameters ($l$, $m$, $r$) that define the triangular membership function. However, the interpretation of the parameters is different in both representations. The vector anchor $\mu$ is equivalent to the middle point of the support $[l, r]$ of the triangular membership function. The vector length $s$ corresponds to the support length, which represents the indeterminacy of the parameter represented. The vector direction is specified by $\gamma$, which is defined as the angle between vertical line $x = \mu$ and the line going through points $(\mu, 0)$ and $(m, 1)$, where $m$ denotes the most possible value of the triangular fuzzy number. The vector representation may also be specified by Cartesian coordinates of its tail and tip points.

We showed that the vector representation is mathematically equivalent to the traditional one and derived formulas for transforming one into the other. In this context, we analysed border cases to illustrate properties of the new representation. We also gave formulas for the addition of two fuzzy parameters and multiplication of a parameter by a crisp number in the vector representation.

Although both representations are defined by three crisp numbers, the difference in their shapes (vector versus triangle) is considerable and may have a high influence on the perception of the underlying information. Triangles, apart from moving along the abscissa, may change the lengths of their three sides and the size of three angles, which does not correspond to any phenomena occurring in everyday life. Accordingly, a change in a single uncertain parameter involves simultaneous changes of seven intertangled features. In a vector only three features may change: fixing point, length, and inclination. Moving points and changing lengths are natural for the human eye to detect. Changes in inclination correspond to the movements of the hands of clocks or meters on the car dashboards, which humans are very accustomed to and comfortable with. For convenience, we have put together the main properties of our proposal and compared them with the corresponding features of the traditional approach in Table 5.

By providing illustrative examples, we show how this new approach may be applied to project management. We considered and compared visualizations of both representations for the duration estimation of a task whose assessment changed from rather optimistic to more and more pessimistic (Example 4) as well as from optimistic to more pessimistic and back to optimistic (Example 5). The underlying problem was the ease of perception of the respective phenomenon by the project manager. It is especially important in the midst of project execution, when they are dealing with dozens of project tasks and have to identify as quickly as possible the problematic ones. It appears that the vector representation is more natural and appealing and, thus, would provide a more efficient (from the point of view of the project manager) visualization of the current situation of individual tasks. The last example (Example 6) concerned the estimation of the project task duration in two possible implementation modes. The underlying problem here was the need to choose one mode for actual project implementation. This is a multicriterial problem. Here the two representations turned out to be complementary: the vector representation visualizes better the difference between the two modes according to some criteria, and the traditional one — according to other ones.

An important soft computing problem needs to be solved if we want to identify a critical path when fuzzy numbers are used to represent project parameters (we mentioned it after Example 6). In such a case, it is necessary to rank (or find the maximum of) fuzzy numbers representing the lengths of several possible paths. This problem is complex and has been subject to extensive research for the traditional representation of fuzzy numbers (e.g., [58,59]). It becomes even more difficult to rank project network paths with fuzzy activity durations (e.g., [54–57]). Often no unique solution exists, and decision makers have to decide arbitrarily which ranking they prefer. The existing ranking methods of fuzzy numbers are strongly determined by their traditional triangular representation — they use notions like centroids, are based on fuzzy number levels, or areas under a selected section of the membership function.

Our new approach for representing fuzzy numbers as vectors provides a completely new perspective for the problem of ranking fuzzy numbers, which should be investigated in the future. We hypothesize that the vector representation may give rise to new fuzzy numbers ranking methods that can be potentially more appropriate in some cases and better reflect the decision maker preferences. It would be interesting to compare the rankings provided by various decision makers for both representations. In the context of project management, this could substantially increase the spectrum of project uncertainty management methods.

In the presented context, vector representations seem to be better and less cognitively demanding than triangles, although in some applications they should be accompanied by the traditional representation. As was demonstrated, vector direction changes are clear and evident at first glance, especially when the angle's sign reverses, which is not the case for changes in the shapes of triangles.

Our previous study on the usability of using vectors as a graphical representation of uncertainty [46] showed the potential usefulness of this approach for visualization purposes in a managerial context. In particular, it appears to be more adapted to the needs of project managers in some practical cases. This preliminary investigation involved scenarios in which static information was presented. It indicated the overall equivalence of both representations in terms of efficiency and effectiveness. This paper's sample visualizations of uncertainty changes over time (Examples 4 and 5) in a project management context suggests

**Table 5**
Properties comparison of traditional and our representations of triangular fuzzy numbers.

| Traditional triangular representation | Our vector representation |
|---|---|
| Graphical form: | |
| Triangle △ | Vector → |
| Numerical form: | |
| Three crisp numbers $(l, m, r)$:<br>$l = \mu - \frac{s}{2}$ (smallest value),<br>$m = \tan(\gamma) + \mu$ (most possible value),<br>$r = \mu + \frac{s}{2}$ (largest value). | Three crisp numbers $(\mu, s, \gamma)$:<br>$\mu = \frac{l+r}{2}$ (middle value),<br>$s = r - l$ (indeterminacy degree),<br>$\gamma = \arctan(m - \mu)$ (pessimistic or optimistic tendency).<br><br>In Cartesian coordinates:<br>Tip $(\mu, 0)$,<br>Tail $(\mu + s \cdot \sin(\gamma), s \cdot \cos(\gamma))$. |
| Graphical features influenced by a change in an uncertain parameter value: | |
| – Location along the abscissa,<br>– Lengths of three triangle sides,<br>– Sizes of three triangle angles. | – Tail location along the abscissa,<br>– Vector length,<br>– Vector direction. |
| Seven components may change, thus making visual processing more challenging and requiring more cognitive effort. | Only three components change, thus simpler and faster visual processing takes place and less cognitive effort is required. |
| Difference between pessimistic or optimistic tendency $(m - \mu)$: | |
| Is measured by distance, which is scale-dependent. | Is measured by angle, which is scale-invariant. |
| Change in pessimistic or optimistic tendency $(m - \mu)$: | |
| The change in the shape of a triangle is not obvious and is visually more difficult to detect. | It is easier to spot the change by observing the vector direction. This may facilitate early identification of forthcoming problems and speed up corrective decisions. |
| Dynamic visualization of uncertain parameter changes in time: | |
| Less obvious interpretation since more graphical features change. | Better suited as they resemble clock hands movements or car meters. |
| Visualization of multiple uncertain parameters: | |
| Processing many triangles is more difficult and graphically awkward as compared to processing multiple vectors. | Displaying many vectors is less visually cluttered and easier to interpret than displaying multiple triangles. They resemble vector fields. |
| Computer implementations and visualizations: | |
| More troublesome compared to vectors from the user interface design point of view. | Easier, since vectors are graphically more compact and take up less space in the graphical user interface. |

that the vector representation may be better suited for presenting the dynamics of uncertainty than triangles. Example 6 proves that a combined approach (triangle, vector) would provide more information for project planning than the traditional representation alone. Moreover, as mentioned in the introduction, some earlier studies on cognitive aspects of information visualization allow us to think that the proposal may have some psychologically based advantages over the traditional representation by triangles [44,45].

In various optimization problems such as scheduling, supply chain management, transportation, it is necessary to perform mathematical operations on fuzzy parameters. In most of those problems, addition and multiplication by a crisp positive number are sufficient. In Section 3.4, we provide appropriate definitions, formulas, and examples of performing these operations in both representations. In this regard, the vector representation is clearly inferior to the traditional one. Although arithmetic of vectors as given by (21) and (32) is not particularly difficult, it involves taking both the tangent and its inverse. This computational complexity provides much more room for rounding error as compared to the traditional version. The classic arithmetic performed on triangular fuzzy numbers is more straightforward, since it involves either three additions of crisp numbers (19) or three multiplications of crisp numbers (31). Thus, it appears advantageous to perform mathematical manipulations on triangular fuzzy numbers using their traditional representations, regardless of how uncertainty data were gathered or graphically presented. We provide the necessary direct formulae for the passage from one representation to the other.

Furthermore, the vector representation also has some specific features that could prevent it from being suitable in some situations. Among them, there is the property of non-linear changes in the $\gamma$ angle with the increase of the distance between the support middle point $\mu$ and the most possible value $m$. This feature will be very useful in project management when the neutrality (neither optimism nor pessimism) of duration estimation means that the middle value of the support is at the same time the most possible value. However, it is not always the case. Therefore, in practical applications, it seems to be reasonable to use both representations. Combining their advantages will provide a fuller picture of parameter uncertainty and its changes over time. We have summarized the key advantages and weaknesses of our proposal in Table 6.

Despite the concerns presented, this research approach provides a new perspective on how to visualize the uncertainty of various parameters. Although we focus in this paper on the project management context, exactly the same representations can be used in any other area. Our proposal may be particularly useful when uncertainty needs to be presented graphically and explored by experts or decision-makers without a formal

**Table 6**
Advantages and weaknesses of our vector representation.

| Advantages | Weaknesses |
|---|---|
| Extends the arsenal of visually representing uncertainty. May be better suited for some groups of users in practical applications [46]. | The new approach, which scientists and practitioners are not familiar with. The need to implement it in existing software that is used for uncertainty visualizations. |
| Strict equivalence with the traditional triangular representation, which allows for interchangeable or simultaneous use. Easy transformations between these representations. | Mathematical operations may be subject to a higher rounding error compared to the traditional version. |
| May be better suited for representing some type of problem, for instance, those in which the detection of the change in the tendency (optimistic/pessimistic) is of the greatest importance. | The maximal absolute value of $\gamma$ angle is smaller than 90° in practical applications. This could be misleading when interpreting the magnitude of an optimistic or pessimistic tendency. |
| Facilitates new research directions. Possible future applications relate to fuzzy number ranking methods. Current solutions are strongly based on geometrical features of the triangular representation. | The change in the $\gamma$ gamma angle is not linearly dependent on $(m-\mu)$. The vector direction is more sensitive closer to the vertical line $x=\mu$ This may be misleading and makes interpretations more complex. |

mathematical background. Extending the arsenal of visualization techniques will allow the decision-makers to define, present, and interpret the uncertain information more accurately. This, in turn, should result in the achievement of specific goals more effectively and efficiently.

The idea of using vectors to visualize uncertainty defined by triangular fuzzy numbers provides further possibilities for scientific exploration. For instance, one may try to search for vector specifications different from those presented in this paper. The presented idea is applicable only to triangular fuzzy numbers, but future research can also be directed to develop other vector representations for different fuzzy number types, e.g., trapezoidal, L-R, intuitionistic, or type-2. It may also be possible to discover ideas for vector-like representations pertaining to other forms of uncertainty than the ones treated in this paper.

## CRediT authorship contribution statement

**Jan Schneider:** Conceptualization, Validation, Formal analysis, Writing – original draft, Writing – review & editing, Visualization. **Dorota Kuchta:** Conceptualization, Validation, Formal analysis, Writing – original draft, Writing – review & editing, Visualization, Funding acquisition. **Rafał Michalski:** Conceptualization, Validation, Formal analysis, Writing – original draft, Writing – review & editing, Visualization, Funding acquisition.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

## Acknowledgements

The work of Dorota Kuchta and Jan Schneider was supported by the Polish National Science Centre, under Grant 394311, 2017/27/B/HS4/01881: "Selected methods supporting project management, taking into consideration various stakeholder groups and using type-2 fuzzy numbers". The work of Rafał Michalski was partially financially supported by the Polish National Science Centre under Grant 2019/35/B/HS4/02892: Intelligent approaches for facility layout problems in management, production, and logistics. We thank the four anonymous reviewers for their constructive comments and suggestions on the previous versions of the paper.

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